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On the Divergence of the Two-Dimensional Dyadic Difference of Dyadic Integrals

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In 1989 F. Schipp and W. R. Wade (*Appl. Anal.* **34**, 203–218) proved for functions in $L(I^2)\log^+ L(I^2)$ (I^2 is the unit square) that the dyadic difference of the dyadic integral $d_n(If)$ converges to f a.e. in the Pringsheim sense (that is, $\min(n_1, n_2)\to\infty$, $n=(n_1, n_2)\in\mathbb{P}^2$). We prove that this result cannot be sharpened. Namely, we prove that for all measurable functions $\delta\colon [0, +\infty)\to[0, +\infty)$, $\lim_{t\to\infty}\delta(t)=0$ we have a function $f\in L\log^+ L\delta(L)$ such as $d_n(If)$ does not converge to f a.e. (in the Pringsheim sense).

In the classical case for the unit square $I^2 := [0, 1)\times[0, 1)$, if g belongs to $L\log^+ L(I^2)$ then

$$g(x, y) = \lim_{h, k\to 0}\frac{1}{hk}\int_x^{x+h}\int_y^{y+k} g(s, t)\,ds\,dt$$

a.e. on I^2 (see Jessen *et al.* [JMZ]). The dyadic analogue of this is proved by Schipp and Wade [SW].

In this paper we give the dyadic analogue of Saks [Sak]; i.e., we prove for all $\delta\colon [0, +\infty)\to[0, +\infty)$ measurable function with property $\lim_{t\to\infty}\delta(t)=0$ the existence of a function $f\in L\log^+ L\delta(L)$ (i.e., $|f(x)|\log^+(|f(x)|)\,\delta(|f(x)|)\in L^1(I^2)$) such as the integral of f on I with respect to both variable is equal to zero and $d_{(n_1, n_2)}(If)$ does not converge to f a.e. as $\min(n_2, n_2)\to\infty$.

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Let $\mathbb{N}$ denote the set of positive integers, $\mathbb{P} := \mathbb{N} \cup \{0\}$, and $I := [0, 1)$. For any set E let E^2 the cartesian product $E \times E$. Thus $\mathbb{P}^2$ is the set of integral lattice points in the first quadrant and I^2 is the unit square. Let $E^1 = E$ and fix $j = 1$ or 2. Denote the j-dimensional Lebesgue measure (μ) of any set $E \subset I^j$ by $\mu(E)$. Denote the $L^p(I^j)$ norm of any function f by $\|f\|_p$ $(1 \leqslant p \leqslant \infty)$.

Denote the dyadic expansion of $n \in \mathbb{P}$ and $x \in I$ by $n = \sum_{j=0}^{\infty} n_j 2^j$ and $x = \sum_{j=0}^{\infty} x_j 2^{-j-1}$ (in the case of $x = k/2^m$ $k, m \in \mathbb{P}$ choose the expansion which terminates in zeros). n_i, x_i are the ith coordinates of n, x, respectively. Set $e_i := 1/2^{i+1} \in I$, the ith coordinate of e_i is 1, the rest are zeros $(i \in \mathbb{P})$. Define the dyadic addition $+$ as

$$x + y = \sum_{j=0}^{\infty} |x_j - y_j| \, 2^{-j-1}.$$

The sets $I_n(x) := \{y \in I : y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\}$ for $x \in I$, $I_n := I_n(0)$ for $n \in \mathbb{N}$ and $I_0(x) := I$ are the dyadic intervals of I. The set of the dyadic intervals on I is denoted by $\mathscr{I} := \{I_n(x) \colon x \in I, n \in \mathbb{P}\}$. Denote by $\mathscr{A}_n$ the σ algebra generated by the sets $I_n(x)$ $(x \in I)$ and E_n the conditional expectation operator with respect to $\mathscr{A}_n$ $(n \in \mathbb{P})$ $(f \in L^1(I))$.

For $t = (t^1, t^2) \in I^2$, $b = (b_1, b_2) \in \mathbb{P}^2$ set the two-dimensional dyadic rectangle, i.e., two-dimensional dyadic interval

$$I_b^2(t) := I_{b_1}(t^1) \times I_{b_2}(t^2).$$

We also use the notation $I_b^2(t) := I_b(t^1) \times I_b(t^2)$ for $b \in \mathbb{P}$, $t = (t^1, t^2)$. For $n = (n_1, n_2) \in \mathbb{P}^2$ denote by $E_n = E_{(n_1, n_2)}$ the two-dimensional expectation operator with respect to the σ algebra $\mathscr{A}_n = \mathscr{A}_{(n_1, n_2)}$ generated by the two-dimensional rectangles $I_{n_1}(x^1) \times I_{n_2}(x^2)$ $(x = (x^1, x^2) \in I^2)$. For $n \in \mathbb{P}$ denote by $|n|$ the greatest integer for which $2^{|n|}$ is not greater than n. That is, $2^{|n|} \leqslant n < 2^{|n|+1}$. The Rademacher functions on I are defined as

$$r_n(x) := (-1)^{x_n} \qquad (x \in I, n \in \mathbb{P}).$$

The Walsh–Paley system (on I) is defined as the set of the Walsh–Paley functions:

$$\omega_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = (-1)^{\sum_{k=0}^{|n|} n_k x_k} \qquad (x \in I, n \in \mathbb{P}).$$

That is, $\omega := (\omega_n, n \in \mathbb{P})$. (For details see Fine [F].) For each function f on I set

$$(d_n f)(t) := \sum_{j=0}^{n-1} 2^{j-1}(f(t) - f(t + e_j))$$

for $t \in I, n \in \mathbb{N}$. Then f is said to be dyadically differentiable at a point $t \in I$ if $(d_n f)(t)$ converges, as $n \to \infty$, to some finite number, $f^{[1]}(t)$ [BW]. Butzer and Wagner [BW] showed that every Walsh function dyadically differentiable with $\omega_k^{[1]}(t) = k\omega_k(t)$ for all $t \in I$ and $k \in \mathbb{P}$.

Let $e_{j_1}^1 := (e_{j_1}, 0)$, $e_{j_2}^2 := (0, e_{j_2}) \in I^2$. For functions of two variables the dyadic difference operator

$$(d_n f)(x) := \sum_{\substack{j_1 < n_1 \\ j_2 < n_2}} 2^{j_1 + j_2 - 2}(f(x) - f(x + e_{j_1}^1) - f(x + e_{j_2}^2) + f(x + e_{j_1}^1 + e_{j_2}^2))$$

$(n = (n_1, n_2) \in \mathbb{N}^2)$ is defined by Schipp and Wade [SW]. The function $f: I^2 \to \mathbb{R}$ is said to be dyadically differentiable at a point $x \in I^2$ if $(d_n f)(x)$ converges, as $\min(n_1, n_2) \to \infty$, to some finite number, $f^{[1]}(x)$ [SW]. Schipp and Wade proved for $\omega_k = \omega_{k_1} \times \omega_{k_2}$ that $\omega_k^{[1]} = k_1 k_2 \omega_k$ $(k = (k_1, k_2) \in \mathbb{P}^2)$.

Let W be the function on I whose Walsh–Fourier coefficients satisfy

$$\hat{W}(k) = \begin{cases} 0 & \text{if } \; k = 0 \\ 1/k & \text{if } \; k \in \mathbb{N}. \end{cases}$$

The dyadic integral of $f: I \to \mathbb{C}$, $f \in L^1$ is defined to be $If := f * W$ [SWS], where $*$ represents dyadic convolution, i.e.,

$$If(t) = \int_I f(t+s)\, W(s)\, d\mu(s) \qquad (t \in I).$$

Schipp [Sch] obtained the differentiation theorem, the following fundamental theorem of dyadic calculus: if $f \in L^1$, $\hat{f}(0) = 0$ then $(If)^{[1]} = f$ a.e. on I. Butzer and Engels defined [BE] the two-dimensional dyadic integral of $f \in L^1(I^2)$ by $If = f * (W \times W)$ where $*$ denotes the two-dimensional dyadic convolution. Schipp and Wade [SW] proved that if $f \in L \log^+ L(I^2)$ and $\hat{f}(n_1, n_2) = 0$ for $n_1 n_2 = 0$ then

$$d_{\mathbf{n}}(If) \to f \qquad \text{as} \quad \min\{n_1, n_2\} \to \infty$$

a.e. on I^2. We prove that this result cannot be sharpened. Namely, we prove

THEOREM 1. *For all measurable function* $\delta: [0, +\infty) \to [0, +\infty)$, $\lim_{t \to \infty} \delta(t) = 0$ *we have a function* $f \in L \log^+ L\delta(L)$ *(this means that*

$$\int_{I^2} |f(x)| \log^+(|f(x)|)\, \delta(f(x))\, d\mu(x) < \infty)$$

with the property

$$\hat{f}(n_1, n_2) = 0 \qquad (n_1 n_2 = 0, n \in \mathbb{P}^2)$$

such as $d_n(If)$ *does not converge to* f *a.e. (in the Pringsheim sense). Moreover,* $\sup_{n \in \mathbb{P}^2} |d_n(If)| = +\infty$ *almost everywhere on* I^2.

In order to prove Theorem 1 we need several lemmas. It is easy to have [SW]

$$d_n(If)(t) = \int_{I^2} f(y^1, y^2)\, d_{n_1} W(t^1 - y^1)\, d_{n_2} W(t^2 - y^2)\, d\mu(y^1, y^2)$$

$$(t = (t^1, t^2) \in I^2, n \in \mathbb{P}^2),$$

where

$$d_n W(x) = D_{2^n}(x) - 1 + \sum_{i=1}^{\infty} \omega_{i2^n}(x) \sum_{s=0}^{2^n - 1} \frac{s\omega_s(x)}{i2^n + s} =: D_{2^n}(x) - 1 + V_n(x).$$

It is also easy to see that

$$V_n(x) = \sum_{i=1}^{\infty} \frac{\omega_{i2^n}(x)}{i} \left(\sum_{s=0}^{2^n - 1} s\omega_s(x) \right) 2^{-n} - \sum_{i=1}^{\infty} \omega_{i2^n}(x) \sum_{s=0}^{2^n - 1} s\omega_s(x) \left(\frac{1}{i2^n} - \frac{1}{i2^n + s} \right)$$
$$=: Z_n(x) - U_n(x)$$

for $x \in I$ and $n \in \mathbb{P}$. The first lemma to be proved

LEMMA 2. *Let* $3 \leqslant n \in \mathbb{P}$ *and* $0 \neq x \in I_{n+3}$. *Then*

$$\sum_{i=1}^{\infty} \frac{\omega_{i2^n}(x)}{i} \geqslant \frac{1}{2},$$

$$d_n W(x) \geqslant 2^{n-1}.$$

Proof of Lemma 2.

$$|U_n(x)| \leqslant \sum_{i=1}^{\infty} \sum_{s=0}^{2^n - 1} s \left| \frac{1}{i2^n} - \frac{1}{i2^n + s} \right|$$
$$\leqslant \sum_{i=1}^{\infty} \frac{1}{i^2 4^n} \sum_{s=0}^{2^n - 1} s^2 \leqslant \sum_{i=1}^{\infty} \frac{2^n}{3i^2}$$
$$= 2^n \frac{\pi^2}{18}.$$

Let $u := 2^n x \pmod 1$, that is, $(u_0 = x_n, u_1 = x_{n+1}, \ldots)$. Since $0 \neq u \in I_3$ then there exists a unique $3 \leqslant t \in \mathbb{P}$ for which $u \in I_t \setminus I_{t+1}$. The Dirichlet kernel [SWS] is

$$D_j(u) = \omega_j(u) \left(\sum_{i=0}^{t-1} j_i 2^i - j_t 2^t \right)$$

and consequently, $|D_j(u)| \leqslant 2^t$. By Abel's transform,

$$\begin{aligned}
\sum_{i=1}^{\infty} \frac{\omega_{i2^n}(x)}{i} &= \sum_{i=1}^{\infty} \frac{\omega_i(u)}{i} \\
&= \sum_{i=1}^{\infty} (D_{i+1}(u) - 1) \left(\frac{1}{i} - \frac{1}{i+1} \right) \\
&= \sum_{i=1}^{2^t-1} i \frac{1}{i(i+1)} + \sum_{i \geqslant 2^t} (D_{i+1}(u) - 1) \frac{1}{i(i+1)} \\
&\geqslant \sum_{i=1}^{2^t-1} i \frac{1}{i(i+1)} - (2^t+1) \sum_{i \geqslant 2^t} \frac{1}{i(i+1)} = \sum_{i=1}^{2^t-1} \frac{1}{i+1} - \frac{2^t+1}{2^t} \\
&\geqslant \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{8} - 1 - \frac{1}{2^3} \geqslant \frac{1}{2}.
\end{aligned}$$

On the other hand,

$$\sum_{s=0}^{2^n-1} s\omega_s(x) = \frac{2^n(2^n-1)}{2}.$$

This gives

$$\begin{aligned}
d_n W(x) &\geqslant 2^n - 1 + 2^{-n} \frac{1}{2} \frac{2^n - 1}{2} 2^n - \frac{\pi^2}{18} 2^n \\
&= 2^n(1 + 1/4 - \pi^2/18) - 5/4 \geqslant 2^n 7/10 - 5/4 \geqslant 2^{n-1} \qquad (n \geqslant 3).
\end{aligned}$$

This completes the proof of Lemma 2. ∎

Define a subset of the set of the two-dimensional intervals $\mathscr{I} \times \mathscr{I}$,

$$\mathscr{I}_{n,a}(x) := \{ I_{n+j}(x^1) \times I_{n+a-j}(x^2) : j = 0, 1, \ldots, a \}$$

for $x \in I^2$, $a, n \in \mathbb{P}$. It is easy to have

$$\bigcap \mathscr{I}_{n,a}(x) = I_{n+a}(x^1) \times I_{n+a}(x^2), \qquad \mu\left(\bigcap \mathscr{I}_{n,a}(x) \right) = 2^{-2n-2a},$$

$F \in \mathscr{I}_{n,a}(x)$ implies $\mu(F) = 2^{-2n-a}$. Next we prove

LEMMA 3. $\mu(\bigcup \mathscr{I}_{n,a}(x)) = (1+a/2)\,2^{-2n-a}$.

Proof. Denote (for the sake of this proof, only)

$$\mu_k := \mu\left(\bigcup_{j=0}^{k} (I_{n+j}(x^1)\times I_{n+a-j}(x^2))\right)$$

for $k=0, 1, \ldots, a$. Then $\mu_0 = 2^{-2n-a}$ and for $k>0$ we have

$$\begin{aligned}
\mu_k &= \mu_{k-1} + \mu(I_{n+k}(x^1)\times I_{n+a-k}(x^2)) \\
&\quad - \mu\left(\bigcup_{j=0}^{k-1} (I_{n+j}(x^1)\times I_{n+a-j}(x^2)) \cap (I_{n+k}(x^1)\times I_{n+a-k}(x^2))\right) \\
&= \mu_{k-1} + \frac{1}{2^{2n+a}} - \mu\left(\bigcup_{j=0}^{k-1} (I_{n+k}(x^1)\times I_{n+a-j}(x^2))\right) \\
&= \mu_{k-1} + \frac{1}{2^{2n+a}} - \mu(I_{n+k}(x^1)\times I_{n+a-k+1}(x^2)) \\
&= \mu_{k-1} + \frac{1}{2^{2n+a}} - \frac{1}{2^{2n+a+1}} = \mu_{k-1} + \frac{1}{2^{2n+a+1}}.
\end{aligned}$$

This gives

$$\begin{aligned}
\mu\left(\bigcup \mathscr{I}_{n,a}(x)\right) &= \mu\left(\bigcup_{j=0}^{a} (I_{n+j}(x^1)\times I_{n+a-j}(x^2))\right) = \mu_a \\
&= \mu_0 + a\frac{1}{2^{2n+a+1}} = \frac{1}{2^{2n+a}} + a\frac{1}{2^{2n+a+1}} = \frac{1+a/2}{2^{2n+a}}.
\end{aligned}$$

This completes the proof of Lemma 3. ▮

Let $b \in \mathbb{P}^{\mathbb{P}}$, $b_0 = 2$, $a \in \mathbb{P}$, and define the sets $J^k_{b,a}$, $\Omega^k_{b,a}$ recursively:

$$J^0_{b,a} := \{0, \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{3}{4}\} \times \{0, \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{3}{4}\},$$

$$\Omega^0_{b,a} := \bigcup_{x \in J^0_{b,a}} \bigcup \mathscr{I}_{b_0, a}(x).$$

Suppose that the sets $J^j_{b,a}$, $\Omega^j_{b,a}$ are defined for $j<k$. Then decompose

$$I^2 \setminus \bigcup_{j=0}^{k-1} \Omega^j_{b,a}$$

as the disjoint union of dyadic squares of the form $I^2_{b_k}(x)$. Take from each dyadic rectangle an element to represent so that the coordinates of which with indices greater than $b_k - 1$ are equal to zero. The set of x's corresponding to these squares is $J^k_{b,a}$. That is,

$$I^2 \setminus \bigcup_{j=0}^{k-1} \Omega^j_{b,a} = \bigcup_{x \in J^k_{b,a}} I^2_{b_k}(x)$$

$(x^1_{b_k} = x^1_{b_k+1} = \cdots = 0,\ x^2_{b_k} = x^2_{b_k+1} = \cdots = 0)$. Then, set

$$\Omega^k_{b,a} := \bigcup_{x \in J^k_{b,a}} \bigcup \mathscr{I}_{b_k,a}(x).$$

Let $b_k > 4(b_{k-1} + a + 1)$ $(k \in \mathbb{N})$. Then sequence b satisfies the equality $b_k \geqslant b_{k-1} + a$ (the two-dimensional dyadic rectangle with the smallest measure in $\Omega^j_{b,a}$ for $j < k$ is of the form $I_{b_{k-1}+a}(x^1) \times I_{b_{k-1}+a}(x^2)$).

This gives the a.e. equality

$$I^2 = \dot{\bigcup_{k=0}^{\infty}} \Omega^k_{b,a} = \dot{\bigcup_{k=0}^{\infty}} \bigcup_{x \in J^k_{b,a}} \bigcup \mathscr{I}_{b_k,a}(x).$$

Let $10 < d \in \mathbb{N}$ be an absolute constant and let $a > 4d$. Set

$$G_{b,a,0} := G_0 := \dot{\bigcup_{k=0}^{\infty}} \bigcup_{x \in J^k_{b,a}} \bigcup \mathscr{I}_{b_k+d+3,\, a-2d}(x) =: \dot{\bigcup_{k=0}^{\infty}} \Omega^k_{b,a,\circ}\,.$$

It is not difficult to prove that

$$\mu(G_0) = \frac{\mu(\bigcup \mathscr{I}_{b_k+d+3,\, a-2d}(0))}{\mu(\bigcup \mathscr{I}_{b_k,a}(0))} = \frac{(1 + (a-2d)/2)\, 2^{-2b_k-6-a}}{(1+a/2)\, 2^{-2b_k-a}} = \frac{1}{2^6}\left(1 - \frac{d}{1+a/2}\right) \geqslant \frac{1}{2^7}.$$

Set for $y \in I^2$

$$f_{b,a}(y) := (-1)^{y^1_0 + y^2_0}\, 2^a \sum_{k=0}^{\infty} \sum_{x \in J^k_{b,a}} (-1)^{y^1_{b_k-1} + y^2_{b_k-1}}\, 1_{I^2_{b_k+a}(x)}(y),$$

where 1_B denotes the characteristic function of any set $B \subset I^2$.

Lemma 4. *For all b, a we have $\int_{I^2} |f_{b,a}| \log^+ |f_{b,a}| \leqslant 2$.*

Proof.

$$\int_{I^2} |f_{b,a}(y)| \log^+(|f_{b,a}(y)|)\, d\mu(y)$$

$$= 2^a \log(2^a) \sum_{k=0}^{\infty} \sum_{x \in J_{b,a}^k} \mu(1_{I^2_{b_k+a}(x)}(y) = 1)$$

$$= 2^a \log(2^a) \sum_{k=0}^{\infty} \sum_{x \in J_{b,a}^k} \mu\left(\bigcap \mathscr{I}_{b_k,a}(x)\right)$$

$$= 2^a \log(2^a) \sum_{k=0}^{\infty} \sum_{x \in J_{b,a}^k} \frac{\mu(\bigcup \mathscr{I}_{b_k,a}(y))}{2^a(1+a/2)}$$

$$\leqslant \frac{\log(2^a)}{1+a/2} \mu(I^2) \leqslant 2.$$

The proof of Lemma 4 is complete. ∎

Since

$$\int_I f_{b,a}(y^1, y^2)\, d\mu(y^1) = \int_I f_{b,a}(y^1, y^2)\, d\mu(y^2) = 0,$$

then for all $n = (n_1, n_2) \in \mathbb{P}^2$, $n_1 n_2 = 0$ we have $\hat{f}_{b,a}(n) = 0$. Consequently, for all $t \in I^2$ and $n \in \mathbb{P}^2$ we have

$$d_n(I f_{b,a})(t) = \int_{I^2} f_{b,a}(y)\, d_{n_1} W(t^1 - y^1)\, d_{n_2} W(t^2 - y^2)\, d\mu(y)$$

$$= \int_{I^2} f_{b,a}(y)((D_{2^{n_1}} + V_{n_1})(t^1 - y^1))((D_{2^{n_2}} + V_{n_2})(t^2 - y^2))\, d\mu(y)$$

$$=: T_n f_{b,a}(t).$$

The following lemma is the very base of the proof of Theorem 1. In the sequel we prove some lemmas which will be necessary in order to prove this basic lemma. The procedure consists of three main steps which will be indicated as cases $\tilde{k} > k$, $\tilde{k} < k$, and $\tilde{k} = k$.

LEMMA 5. *Let $t \in G_0$. Then there exists an $n \in \mathbb{P}^2$ (the exact form of n see below) for which*

$$|T_n f_{b,a}(t)| \geqslant 2^{-4}.$$

Let $t \in G_0$. Then there exists a unique $k \in \mathbb{P}$, $x \in J_{b,a}^k$ for which

$$t \in \bigcup \mathscr{I}_{b_k+d+3,\, a-2d}(x).$$

Hence also exists a $j \in \{d, d+1, ..., a-d\}$ such that

$$t \in I_{b_k+3+j}(x^1) \times I_{b_k+3+a-j}(x^2).$$

Let $n=(n_1, n_2)=(b_k+j, b_k+a-j)$. For $y \in I^2_{b_k+a}(x)$ we have $t-y \in I_{n_1+3} \times I_{n_2+3}$. By Lemma 2 it follows

$$\begin{aligned}
&\left|\int_{I^2_{b_k}(x)} f_{b,a}(y)\, d_{n_1}W(t^1-y^1)\, d_{n_2}W(t^2-y^2)\, d\mu(y)\right| \\
&= \left|\int_{I^2_{b_k+a}(x)} f_{b,a}(y)\, d_{n_1}W(t^1-y^1)\, d_{n_2}W(t^2-y^2)\, d\mu(y)\right| \\
&= \left|\int_{I^2_{b_k+a}(x)} 2^a(-1)^{y^1_0+y^2_0+y^1_{b_k-1}+y^2_{b_k-1}}\, d_{n_1}W(t^1-y^1)\, d_{n_2}W(t^2-y^2)\, d\mu(y)\right| \\
&\geqslant \left|\int_{I^2_{b_k+a}(x)} 2^a 2^{n_1+n_2-2} d\mu(y)\right| \\
&= 2^{-2b_k-2a+a+n_1+n_2-2} \geqslant \tfrac{1}{4}.
\end{aligned}$$

In order to prove Lemma 5 we give an upper bound for the integral

$$\left|\int_{I^2 \backslash I^2_{b_k+a}(x)} f_{b,a}(y)\, d_{n_1}W(t^1-y^1)\, d_{n_2}W(t^2-y^2)\, d\mu(y)\right|$$

for $t \in I_{b_k+3+j}(x^1) \times I_{b_k+3+a-j}(x^2)$, where $j \in \{d, d+1, ..., a-d\}$.

Lemma 6. *The case $\tilde{k}<k$. We prove*

$$\begin{aligned}
\int_{\cup_{\tilde{k}<k}\Omega^{\tilde{k}}_{b,a}} & f_{b,a}(y)(D_{2^{n_1}}(t^1-y^1)+V_{n_1}(t^1-y^1)) \\
&\times (D_{2^{n_2}}(t^2-y^2)+V_{n_2}(t^2-y^2))\, d\mu(y)=0.
\end{aligned}$$

Proof. Let $y \in \Omega^{\tilde{k}}_{b,a}$ for some $\tilde{k}<k$. Then there exists a unique $\tilde{x} \in J^{\tilde{k}}_{b,a}$ for which $y \in I^2_{b_{\tilde{k}}+a}(\tilde{x})$ (otherwise $f_{b,a}(y)=0$). Then for any $i_1 \in \mathbb{N}$

$$\begin{aligned}
&\int_{I_{b_{\tilde{k}}+a}(\tilde{x}^1)} f_{b,a}(y)\, \omega_{i_1 2^{n_1}}(t^1-y^1)\, d\mu(y^1) \\
&\quad = (-1)^{\tilde{x}^1_0+\tilde{x}^2_0+\tilde{x}^1_{b_k-1}+\tilde{x}^2_{b_k-1}}\, 2^a \int_{I_{b_{\tilde{k}}+a}(\tilde{x}^1)} \omega_{i_1 2^{n_1}}(t^1-y^1)\, d\mu(y^1)=0
\end{aligned}$$

since $b_{\tilde{k}}+a<b_{\tilde{k}+1}\leqslant b_k<n_1$. Thus,

$$\int_{I_{b_{\tilde{k}}+a}(\tilde{x}^1)} f_{b,a}(y)\, V_{n_1}(t^1-y^1)\, d\mu(y^1)=0.$$

Similarly,

$$\int_{I_{b_{\tilde{k}}+a}(\tilde{x}^2)} f_{b,a}(y)\, V_{n_2}(t^2-y^2)\, d\mu(y^2)=0.$$

Since $y\in I^2_{b_{\tilde{k}}+a}(\tilde{x})$ and $t\in I^2_{b_k}(x)$, then $t-y\notin I^2_{b_k}$ and consequently, either $t^1-y^1\notin I_{b_k}$ or $t^2-y^2\notin I_{b_k}$. That is, we have

$$D_{2^{n_1}}(t^1-y^1)\cdot D_{2^{n_2}}(t^2-y^2)=0.$$

This and the above implies

$$\int_{I^2_{b_{\tilde{k}}+a}(\tilde{x})} f_{b,a}(y)(D_{2^{n_1}}(t^1-y^1)+V_{n_1}(t^1-y^1)) \times (D_{2^{n_2}}(t^2-y^2)+V_{n_2}(t^2-y^2))\, d\mu(y)=0.$$

That is, we have

$$\int_{\cup_{\tilde{k}<k}\Omega^{\tilde{k}}_{b,a}} f_{b,a}(y)(D_{2^{n_1}}(t^1-y^1)+V_{n_1}(t^1-y^1)) \times (D_{2^{n_2}}(t^2-y^2)+V_{n_2}(t^2-y^2))\, d\mu(y)=0.$$

This completes the proof of Lemma 6. ∎

LEMMA 7. *The case $\tilde{k}>k$,*

$$\left|\int_{\cup_{\tilde{k}>k}\Omega^{\tilde{k}}_{b,a}} f_{b,a}(y)(D_{2^{n_1}}(t^1-y^1)+V_{n_1}(t^1-y^1)) \times (D_{2^{n_2}}(t^2-y^2)+V_{n_2}(t^2-y^2))\, d\mu(y)\right|\leqslant 2^{-5}.$$

Proof. Let $\tilde{k}>k$, $\tilde{x}\in J^{\tilde{k}}_{b,a}$. Then

$$\tilde{x}=(\tilde{x}^1,\tilde{x}^2),\ (\tilde{x}^1+e_{b_{\tilde{k}}-1},\tilde{x}^2),\ (\tilde{x}^1,\tilde{x}^2+e_{b_{\tilde{k}}-1}),\ (\tilde{x}^1+e_{b_{\tilde{k}}-1},\tilde{x}^2+e_{b_{\tilde{k}}-1})\in J^{\tilde{k}}_{b,a}.$$

Since $n=(b_k+j,\, b_k+a-j)\leqslant(b_k+a,\, b_k+a)<(b_{\tilde{k}}-1,\, b_{\tilde{k}}-1)$ then denoting $(\tilde{x}+\varepsilon e_{b_{\tilde{k}}-1})=(\tilde{x}^1+\varepsilon_1 e_{b_{\tilde{k}}-1},\, \tilde{x}^2+\varepsilon_2 e_{b_{\tilde{k}}-1})$ as $\varepsilon_1, \varepsilon_2=0, 1$ we have

$$\sum_{\varepsilon_1, \varepsilon_2=0, 1} \int_{I^2_{b_{\tilde{k}}+a}(\tilde{x}+\varepsilon e_{b_{\tilde{k}}-1})} f_{b, a}(y)\left(D_{2^{n_1}}(t^1-y^1)+\sum_{i_1=1}^{2^{b_{\tilde{k}}-n_1-2}}\sum_{s=0}^{2^{n_1}-1}\frac{\omega_{i_1 2^{n_1}+s}(t^1-y^1)\, s}{i_1 2^{n_1}+s}\right)$$
$$\times(D_{2^{n_2}}(t^2-y^2)+V_{n_2}(t^2-y^2))\, d\mu(y)$$
$$=(-1)^{\tilde{x}^1_0+\tilde{x}^2_0+\tilde{x}^1_{b_k-1}+\tilde{x}^2_{b_k-1}}\, 2^a \sum_{\varepsilon_1, \varepsilon_2=0, 1}(-1)^{\varepsilon_1+\varepsilon_2}2^a$$
$$\times\left(D_{2^{n_1}}(t^1-\tilde{x}^1)+\sum_{i_1=1}^{2^{b_{\tilde{k}}-n_1-2}}\sum_{s=0}^{2^{n_1}-1}\frac{\omega_{i_1 2^{n_1}+s}(t^1-\tilde{x}^1)\, s}{i_1 2^{n_1}+s}\right)$$
$$\times\int_{I^2_{b_{\tilde{k}}+a}(\tilde{x}+\varepsilon e_{b_{\tilde{k}}-1})}(D_{2^{n_2}}(t^2-y^2)+V_{n_2}(t^2-y^2))\, d\mu(y)=0,$$

because

$$\sum_{\varepsilon_1, \varepsilon_2=0, 1}(-1)^{\varepsilon_1+\varepsilon_2}\int_{I^2_{b_{\tilde{k}}+a}(\tilde{x}+\varepsilon e_{b_{\tilde{k}}-1})}(D_{2^{n_2}}(t^2-y^2)+V_{n_2}(t^2-y^2))\, d\mu(y)=0$$

for both $\varepsilon_2=0$ and $\varepsilon_2=1$ since $D_{2^{n_2}}(t^2-y^2)+V_{n_2}(t^2-y^2)$ does not depend on ε_1. Similarly,

$$\sum_{\varepsilon_1, \varepsilon_2=0, 1}\int_{I^2_{b_{\tilde{k}}+a}(\tilde{x}+\varepsilon e_{b_{\tilde{k}}-1})} f_{b, a}(y)(D_{2^{n_1}}(t^1-y^1)+V_{n_1}(t^1-y^1))$$
$$\times\left(D_{2^{n_2}}(t^2-y^2)+\sum_{i_2=1}^{2^{b_{\tilde{k}}-n_2-2}}\sum_{s=0}^{2^{n_2}-1}\frac{\omega_{i_2 2^{n_2}+s}(t^2-y^2)\, s}{i_2 2^{n_2}+s}\right)d\mu(y)=0.$$

On the other hand,

$$\int_{I^2_{b_{\tilde{k}}+a}(\tilde{x})} f_{b, a}(y)\left(\sum_{i_1=2^{b_{\tilde{k}}-n_1+a}}^{\infty}\sum_{s=0}^{2^{n_1}-1}\frac{\omega_{i_1 2^{n_1}+s}(t^1-y^1)\, s}{i_1 2^{n_1}+s}\right)$$
$$\times(D_{2^{n_2}}(t^2-y^2)+V_{n_2}(t^2-y^2))\, d\mu(y)$$
$$=(-1)^{\tilde{x}^1_0+\tilde{x}^2_0+\tilde{x}^1_{b_k-1}+\tilde{x}^2_{b_k-1}}\, 2^a \sum_{i_1=2^{b_{\tilde{k}}-n_1+a}}^{\infty}\sum_{s=0}^{2^{n_1}-1}\frac{\omega_s(t^1-\tilde{x}^1)\, s}{i_1 2^{n_1}+s}$$
$$\times\int_{I^2_{b_{\tilde{k}}+a}(\tilde{x})}\omega_{i_1 2^{n_1}}(t^1-y^1)(D_{2^{n_2}}(t^2-y^2)+V_{n_2}(t^2-y^2))\, d\mu(y^1)\, d\mu(y^2)=0,$$

because

$$\int_{I_{b_{\tilde{k}}+a}(\tilde{x}^1)} \omega_{i_1 2^{n_1}}(t^1 - y^1)\, d\mu(y^1) = 0$$

for $i_1 \geqslant 2^{b_{\tilde{k}} - n_1 + a}$. Similarly, we have

$$\int_{I^2_{b_{\tilde{k}}+a}(\tilde{x})} f_{b,a}(y)(D_{2^{n_1}}(t^1 - y^1) + V_{n_1}(t^1 - y^1)) \times \left(\sum_{i_2 = 2^{b_{\tilde{k}} - n_2 + a}}^{\infty} \sum_{s=0}^{2^{n_2}-1} \frac{\omega_{i_2 2^{n_2}+s}(t^2 - y^2)\, s}{i_2 2^{n_2} + s} \right) d\mu(y) = 0.$$

Consequently, in the case of $\tilde{k} > k$,

(1)
$$\begin{aligned} &\int_{\bigcup_{\tilde{k}>k} \Omega^{\tilde{k}}_{b,a}} f_{b,a}(y)(D_{2^{n_1}}(t^1 - y^1) + V_{n_1}(t^1 - y^1)) \\ &\quad \times (D_{2^{n_2}}(t^2 - y^2) + V_{n_2}(t^2 - y^2))\, d\mu(y) \\ &= \sum_{\tilde{k}=k+1}^{\infty} \sum_{\tilde{x} \in J^{\tilde{k}}_{b,a}} 2^a (-1)^{\tilde{x}^1_0 + \tilde{x}^2_0 + \tilde{x}^1_{b_k - 1} + \tilde{x}^2_{b_k - 1}} \\ &\quad \times \int_{I^2_{b_{\tilde{k}}+a}(\tilde{x})} \left[\sum_{i_1 = 2^{b_{\tilde{k}} - n_1 - 2}}^{2^{b_{\tilde{k}} - n_1 + a}} \sum_{s=0}^{2^{n_1}-1} \frac{\omega_{i_1 2^{n_1}+s}(t^1 - y^1)\, s}{i_1 2^{n_1} + s} \right] \\ &\quad \times \left[\sum_{i_2 = 2^{b_{\tilde{k}} - n_2 - 2}}^{2^{b_{\tilde{k}} - n_2 + a}} \sum_{s=0}^{2^{n_2}-1} \frac{\omega_{i_2 2^{n_2}+s}(t^2 - y^2)\, s}{i_2 2^{n_2} + s} \right] d\mu(y). \end{aligned}$$

Let $n, i \in \mathbb{N}$. Then

$$\sum_{s=0}^{2^n - 1} \left| \frac{\omega_s s}{i 2^n + s} - \frac{\omega_s s}{i 2^n} \right| \leqslant \sum_{s=0}^{2^n-1} \frac{s^2}{i^2 4^n} \leqslant \frac{2^n}{i^2}.$$

It is easy to have $\sum_{i=l}^{L} \frac{1}{2i^2} \leqslant 1/l$. By Abel's transform it follows

$$\begin{aligned} &\sum_{i=l}^{L} \frac{\omega_{i 2^n}}{i} \left(\sum_{s=0}^{2^n-1} \omega_s s 2^{-n} \right) \\ &\quad = \sum_{s=0}^{2^n-1} \omega_s s 2^{-n} \left[\sum_{i=l}^{L} \left(\sum_{j=l}^{i} \omega_{j 2^n} \frac{1}{i(i+1)} \right) + \sum_{j=l}^{L} \omega_{j 2^n} \frac{1}{L+1} \right]. \end{aligned}$$

Since $\|\sum_{j=l}^{i} \omega_{j2^n}\|_1 = \|D_{i+1}(2^n\cdot)-D_l(2^n\cdot)\|_1 \leqslant \log_2(L+1)+\log_2(l) \leqslant 4\log(L)$ (for $L \geqslant 1$) and

$$\left\| \sum_{i=l}^{L} \frac{\omega_{i2^n}}{i} \left(\sum_{s=0}^{2^n-1} \omega_s s 2^{-n} \right) \right\|_1 \leqslant 2^{n+2} \log(L)/l$$

then for the absolute value of (1) we get the following upper bound (apply that $b_{\tilde{k}} > 4b_k+4a+4$ for $\tilde{k} > k$)

$$\begin{aligned}
&\left| \sum_{\tilde{k}=k+1}^{\infty} \sum_{\tilde{x} \in J_{b,a}^{\tilde{k}}} 2^a \int_{I^2_{b_{\tilde{k}}+a}(\tilde{x})} \left[\sum_{i_1=2^{b_{\tilde{k}}-n_1-2}}^{2^{b_{\tilde{k}}-n_1+a}} \sum_{s=0}^{2^{n_1}-1} \frac{\omega_{i_1 2^{n_1}+s}(t^1-y^1)\, s}{i_1 2^{n_1}+s} \right] \right. \\
&\quad \left. \times \left[\sum_{i_2=2^{b_{\tilde{k}}-n_2-2}}^{2^{b_{\tilde{k}}-n_2+a}} \sum_{s=0}^{2^{n_2}-1} \frac{\omega_{i_2 2^{n_2}+s}(t^2-y^2)\, s}{i_2 2^{n_2}+s} \right] \right| d\mu(y) \\
&\leqslant \sum_{\tilde{k}=k+1}^{\infty} 2^a \int_{I^2} \left| \sum_{i_1=2^{b_{\tilde{k}}-n_1-2}}^{2^{b_{\tilde{k}}-n_1+a}} \sum_{s=0}^{2^{n_1}-1} \frac{\omega_{i_1 2^{n_1}+s}(t^1-y^1)\, s}{i_1 2^{n_1}+s} \right. \\
&\quad \left. \times \sum_{i_2=2^{b_{\tilde{k}}-n_2-2}}^{2^{b_{\tilde{k}}-n_2+a}} \sum_{s=0}^{2^{n_2}-1} \frac{\omega_{i_2 2^{n_2}+s}(t^2-y^2)\, s}{i_2 2^{n_2}+s} \right| d\mu(y) \\
&\leqslant \sum_{\tilde{k}=k+1}^{\infty} 2^a \left[\frac{2^{n_1}}{2^{b_{\tilde{k}}-n_1-2}} + 2^{n_1+2} \frac{\log(2^{b_{\tilde{k}}-n_1+a})}{2^{b_{\tilde{k}}-n_1-2}} \right] \\
&\quad \times \left[\frac{2^{n_2}}{2^{b_{\tilde{k}}-n_2-2}} + 2^{n_2+2} \frac{\log(2^{b_{\tilde{k}}-n_2+a})}{2^{b_{\tilde{k}}-n_2-2}} \right] \\
&\leqslant \sum_{\tilde{k}=k+1}^{\infty} 2^a \left[\frac{2^{b_k+a}}{2^{b_{\tilde{k}}-b_k-a}} + 2^{b_k+a} \frac{\log(2^{b_{\tilde{k}}})}{2^{b_{\tilde{k}}-b_k-a}} \right] \\
&\quad \times \left[\frac{2^{b_k+a}}{2^{b_{\tilde{k}}-b_k-a}} + 2^{b_k+a} \frac{\log(2^{b_{\tilde{k}}})}{2^{b_{\tilde{k}}-b_k-a}} \right] \\
&\leqslant \sum_{\tilde{k}=k+1}^{\infty} 2^a \left[\frac{1}{2^{b_{\tilde{k}}/2}} + 2^{b_k+a} \frac{b_{\tilde{k}}}{2^{3b_{\tilde{k}}/4}} \right]^2 \\
&\leqslant \sum_{\tilde{k}=k+1}^{\infty} 2^a \left[\frac{b_{\tilde{k}}}{2^{b_{\tilde{k}}/2}} \right]^2 < 2^{-5}
\end{aligned}$$

(recall that sequence b is strictly monotone increasing, and $b_1 > 4(b_0+a+1) > 172$). This completes the proof of Lemma 7. ∎

Next, we discuss the case $\tilde{k} = k$. Let $x \in J_{b,a}^{k}$ and

$$t \in I_{b_k+3+j}(x^1) \times I_{b_k+3+a-j}(x^2),$$

where $j \in \{d, d+1, \ldots, a-d\}$ and let $n_1 := 2^{b_k+j}$. Set

$$\Omega_{b,a}^{k,1} := \bigcup_{x \in J_{b,a}^k} I_{b_k}(x^1).$$

LEMMA 8. *We prove*

$$\left| \int_{\Omega_{b,a}^{k,1} \setminus I_{b_k+a}(x^1)} f_{b,a}(y)\, U_{n_1}(t^1 - y^1)\, d\mu(y^1) \right| \leqslant 130.$$

Proof. In order to save space denote n_1 by simply n—only in the proof of this lemma (note that in the paper $n = (n_1, n_2) \in \mathbb{P}^2$ generally).

$$\begin{aligned} U_n(x) &= \sum_{i=1}^{\infty} \omega_{i2^n}(x) \sum_{s=0}^{2^n-1} s\omega_s(x) \left(\frac{1}{i2^n} - \frac{1}{i2^n+s} \right) \\ &= \sum_{i=1}^{\infty} \frac{\omega_{i2^n}(x)}{i2^n} \sum_{s=0}^{2^n-1} \frac{s^2 \omega_s(x)}{i2^n+s}. \end{aligned}$$

For $s = s_0 + s_1 2^1 + \cdots + s_{n-1} 2^{n-1}$ let $l := 1 - s_0 + (1-s_1)\, 2^1 + \cdots + (1-s_{n-1})\, 2^{n-1}$. That is, $s + l = 2^n - 1$.

$$\omega_{2^n-1-l} = \omega_s = r_0^{s_0} \ldots r_{n-1}^{s_{n-1}} = r_0 r_0^{1-s_0} \ldots r_{n-1} r_{n-1}^{1-s_{n-1}} = r_0 \ldots r_{n-1} \omega_l = \omega_{2^n-1} \omega_l.$$

Then

$$\begin{aligned} \sum_{s=0}^{2^n-1} \frac{s^2 \omega_s}{i2^n+s} &= \sum_{l=0}^{2^n-1} \frac{(2^n-1-l)^2\, \omega_{2^n-1-l}}{(i+1)\, 2^n - 1 - l} \\ &= \omega_{2^n-1} \left((2^n-1)^2 \sum_{l=0}^{2^n-1} \frac{\omega_l}{(i+1)\, 2^n - 1 - l} \right. \\ &\quad \left. - 2(2^n-1) \sum_{l=0}^{2^n-1} \frac{l\omega_l}{(i+1)\, 2^n - 1 - l} + \sum_{l=0}^{2^n-1} \frac{l^2 \omega_l}{(i+1)\, 2^n - 1 - l} \right). \end{aligned}$$

Denote by $f^{[u]}$ the uth dyadic derivative of the function f ($u \in \mathbb{P}$, $f^{[0]} := f$),

$$\begin{aligned} \sum_{l=0}^{2^n-1} \frac{\omega_l}{(i+1)\, 2^n - 1 - l} &= \sum_{l=0}^{2^n-1} \omega_l \frac{\dfrac{1}{(i+1)\, 2^n - 1}}{1 - \dfrac{l}{(i+1)\, 2^n - 1}} \\ &= \sum_{l=0}^{2^n-1} \omega_l \frac{1}{(i+1)\, 2^n - 1} \sum_{u=0}^{\infty} \left[\frac{l}{(i+1)\, 2^n - 1} \right]^u \end{aligned}$$

$$=\frac{1}{(i+1)\,2^n-1}\sum_{u=0}^{\infty}\left[\frac{1}{(i+1)\,2^n-1}\right]^u\sum_{l=0}^{2^n-1} l^u\omega_l$$

$$=\frac{1}{(i+1)\,2^n-1}\sum_{u=0}^{\infty}\left[\frac{1}{(i+1)\,2^n-1}\right]^u D_{2^n}^{[u]}.$$

In a similar way we also have

$$\sum_{l=0}^{2^n-1}\frac{l\omega_l}{(i+1)\,2^n-1-l}=\sum_{l=0}^{2^n-1}\omega_l\,\frac{\dfrac{l}{(i+1)\,2^n-1}}{1-\dfrac{l}{(i+1)\,2^n-1}}$$

$$=\sum_{l=0}^{2^n-1}\omega_l\sum_{u=1}^{\infty}\left[\frac{l}{(i+1)\,2^n-1}\right]^u$$

$$=\sum_{u=1}^{\infty}\left[\frac{1}{(i+1)\,2^n-1}\right]^u D_{2^n}^{[u]}$$

and

$$\sum_{l=0}^{2^n-1}\frac{l^2\omega_l}{(i+1)\,2^n-1-l}=((i+1)\,2^n-1)\sum_{u=2}^{\infty}\left[\frac{1}{(i+1)\,2^n-1}\right]^u D_{2^n}^{[u]}.$$

That is,

$$\sum_{s=0}^{2^n-1}\frac{s^2\omega_s}{i2^n+s}=\omega_{2^n-1}\left(\frac{(2^n-1)^2}{(i+1)\,2^n-1}\sum_{u=0}^{\infty}\left[\frac{1}{(i+1)\,2^n-1}\right]^u D_{2^n}^{[u]}\right.$$

$$-2(2^n-1)\sum_{u=1}^{\infty}\left[\frac{1}{(i+1)\,2^n-1}\right]^u D_{2^n}^{[u]}$$

$$\left.+((i+1)\,2^n-1)\sum_{u=2}^{\infty}\left[\frac{1}{(i+1)\,2^n-1}\right]^u D_{2^n}^{[u]}\right)$$

$$=\omega_{2^n-1}\left(\frac{(2^n-1)^2}{(i+1)\,2^n-1}D_{2^n}+\left(\frac{(2^n-1)^2}{((i+1)\,2^n-1)^2}-\frac{2(2^n-1)}{(i+1)\,2^n-1}\right)D_{2^n}^{[1]}\right.$$

$$+\left(\frac{(2^n-1)^2}{(i+1)\,2^n-1}+(i+1)\,2^n-1-2(2^n-1)\right)$$

$$\left.\times\sum_{u=2}^{\infty}\left[\frac{1}{(i+1)\,2^n-1}\right]^u D_{2^n}^{[u]}\right).$$

We integrate $f_{b,a}(y)$ on

$$\bigcup_{\substack{\tilde{x}\in J_{b,a}^k\\ \tilde{x}^1\neq x^1}} I_{b_k+a}(\tilde{x}^1)$$

because $\Omega^{k,1}_{b,a}\setminus I_{b_k}(x^1)=\bigcup_{\substack{\tilde{x}\in J^k_{b,a}\\ \tilde{x}^1\neq x^1}} I_{b_k}(\tilde{x}^1)$ but if $y^1\in I_{b_k}(x^1)\setminus I_{b_k+a}(x^1)$ then $f_{b,a}(y)=0$ for all $x\in J^k_{b,a}$.

$t^1\in I_n(x^1)$ (remark that in this proof (only, not elsewhere) $n_1=n$) which gives $D_{2^n}(t^1-y^1)=0$. Discuss $D^{[u]}_{2^n}$. By induction we have

$$\begin{aligned} D^{[u]}_{2^n}(z) &= \sum_{s_1=0}^{n-1}\cdots\sum_{s_u=0}^{n-1} 2^{s_1+\cdots+s_u-u}\sum_{\varepsilon_1,\ldots,\varepsilon_u\in\{0,1\}}(-1)^{\varepsilon_1+\cdots+\varepsilon_u}\\ &\quad\times D_{2^n}(z+\varepsilon_1 e_{s_1}+\cdots+\varepsilon_u e_{s_u})\\ &= \sum_{s\in\{0,1,\ldots,n-1\}^u} 2^{s\cdot 1-u}\sum_{\varepsilon\in\{0,1\}^u}(-1)^{\varepsilon\cdot 1} D_{2^n}(z+\varepsilon e_s). \end{aligned}$$

For a given $s\in\{0,1,\ldots,n-1\}^u$ and $\varepsilon\in\{0,1\}^u$ there exists at most one $\tilde{x}^1\in I$ for which there exists an $\tilde{x}=(\tilde{x}^1,\tilde{x}^2)\in J^k_{b,a}$ for which $t^1-\tilde{x}^1+\varepsilon e_s\in I_{b_k}$, that is, for this s and ε we have

$$\begin{aligned} &\int_{I_{b_k}(\tilde{x})} |f_{b,a}(y)|\, D_{2^n}(t^1-\tilde{x}^1+\varepsilon e_s)\, d\mu(y^1)\\ &\quad=\int_{I_{b_k+a}(\tilde{x})} |f_{b,a}(y)|\, D_{2^n}(t^1-\tilde{x}^1+\varepsilon e_s)\, d\mu(y^1)\leqslant 2^a 2^n 2^{-b_k-a}=2^{n-b_k}. \end{aligned}$$

Consequently,

$$\begin{aligned} &\left|\int_{\Omega^{k,1}_{b,a}\setminus I_{b_k+a}(x^1)} f_{b,a}(y)\, U_n(t^1-y^1)\, d\mu(y^1)\right|\\ &\quad\leqslant \sum_{i=1}^{\infty}\left|\int_{\Omega^{k,1}_{b,a}\setminus I_{b_k+a}(x^1)} f_{b,a}(y)\,\frac{\omega_{i2^n}(t^1-y^1)}{i2^n}\,\omega_{2^n-1}(t^1-y^1)\right.\\ &\quad\times\left[\left(\frac{(2^n-1)^2}{((i+1)\,2^n-1)^2}-\frac{2(2^n-1)}{(i+1)\,2^n-1}\right)\right.\\ &\quad\times\sum_{s=0}^{n-1}\left(D_{2^n}(t^1-y^1)-D_{2^n}(t^1-y^1+e_s)\right)2^{s-1}\\ &\quad+\left(\frac{(2^n-1)^2}{(i+1)\,2^n-1}+(i+1)\,2^n-1-2(2^n-1)\right)\\ &\quad\times\left.\left.\sum_{u=2}^{\infty}\frac{1}{((i+1)\,2^n-1)^u}\,D^{[u]}_{2^n}(t^1-y^1)\right]d\mu(y^1)\right| \end{aligned}$$

$$\leqslant \sum_{i=1}^{\infty} \sum_{s_1=0}^{n-1} \left[\int_{\Omega_{b,a}^{k,1} \setminus I_{b_k+a}(x^1)} |f_{b,a}(y)| \, D_{2^n}(t^1 - y^1 + e_{s_1}) \, d\mu(y^1) \right]$$

$$\times \frac{2^{s_1-1}}{i2^n} \left(\frac{(2^n-1)^2}{((i+1)\,2^n-1)^2} + \frac{2(2^n-1)}{(i+1)\,2^n-1} \right)$$

$$+ \sum_{i=1}^{\infty} \sum_{u=2}^{\infty} \frac{1}{i2^n} \left(\frac{(2^n-1)^2}{(i+1)\,2^n-1} + (i+1)\,2^n + 1 + 2(2^n-1) \right)$$

$$\times \frac{1}{((i+1)\,2^n-1)^u} \sum_{s \in \{0,1,\ldots,n-1\}^u} \sum_{\varepsilon \in \{0,1\}^u} 2^{s \cdot 1 - u}$$

$$\times \int_{\Omega_{b,a}^{k,1} \setminus I_{b_k+a}(x^1)} |f_{b,a}(y)| \, D_{2^n}(t^1 - y^1 + \varepsilon e_s) \, d\mu(y^1) =: (3.1) + (3.2).$$

If there is no s_i (or s_1 in the case of (3.1)) for which $s_i < b_k$, then $t^1 \in I_{b_k}(x^1)$, $y^1 \in I_{b_k}(\tilde{x}^1)$ $(\tilde{x}^1 \neq x^1)$ implies $t^1 - y^1 \notin I_{b_k}$, and consequently, $t^1 - y^1 + \varepsilon e_s \notin I_{b_k}$. Which gives $D_{2^n}(t^1 - y^1 + \varepsilon e_s) = 0$. That is, if we take account the addends in (3.1) and (3.2) which differ from zero, we have to suppose that there is an $i \in \{1, \ldots, u\}$ for which $s_i < b_k$. Since

$$\sum_{s_1=0}^{b_k-1} \sum_{s_2=0}^{n-1} \cdots \sum_{s_u=0}^{n-1} \sum_{\varepsilon \in \{0,1\}^u} 2^{s \cdot 1 - u} \leqslant 2^{b_k} 2^{(u-1)n},$$

then

$$\sum_{\{s \in \{0,1,\ldots,n-1\}^u : \exists s_i < b_k\}} \sum_{\varepsilon \in \{0,1\}^u} 2^{s \cdot 1 - u} \leqslant u 2^{b_k + (u-1)n}.$$

This gives a bound for (3.1) as

$$\sum_{i=1}^{\infty} \frac{1}{i2^n} \left(\frac{(2^n-1)^2}{((i+1)\,2^n-1)^2} + \frac{2(2^n-1)}{(i+1)\,2^n-1} \right) 2^{b_k} \frac{1}{2^{b_k+a}} 2^a 2^n \leqslant \frac{\pi^2}{2}.$$

For (3.2) we get the following upper bound

$$\sum_{i=1}^{\infty} \sum_{u=2}^{\infty} \frac{1}{i2^n} \left(\frac{(2^n-1)^2}{(i+1)\,2^n-1} + (i+1)\,2^n + 1 + 2(2^n-1) \right)$$

$$\times \frac{1}{((i+1)\,2^n-1)^u} u 2^{b_k + (u-1)n} \frac{1}{2^{b_k+a}} 2^a 2^n$$

$$= \sum_{i=1}^{\infty} \frac{1}{i2^n} \left(\frac{(2^n-1)^2}{(i+1)\,2^n-1} + (i+1)\,2^n + 1 + 2(2^n-1) \right) \sum_{u=2}^{\infty} \frac{u 2^{nu}}{((i+1)\,2^n-1)^u}.$$

Let $x := 2^n/((i+1)\,2^n-1) = 1/(i+1-1/2^n) < 1$ $(i \geqslant 1)$. Then

$$\sum_{u=0}^{\infty} x^u = \frac{1}{1-x}, \qquad \sum_{u=1}^{\infty} u x^u = \frac{x}{(1-x)^2}.$$

Since $n \geqslant 2$ then

$$\sum_{u=2}^{\infty} u x^u = \frac{x}{(1-x)^2} - x = x\left(\frac{1}{(1-x)^2} - 1\right)$$

$$= x^2 \frac{2-x}{(1-x)^2} \leqslant x^2 \frac{2 - \dfrac{1}{1+1-1/2}}{\left(1 - \dfrac{1}{1+1-1/2}\right)^2} = 12x^2.$$

Since $x^2 \leqslant 1/i^2$ then for (3.2) we have the following upper bound

$$12 \sum_{i=1}^{\infty} \frac{1}{i^3 2^n}\left(\frac{(2^n-1)^2}{(i+1)\,2^n-1} + (i+1)\,2^n + 1 + 2(2^n-1)\right)$$

$$\leqslant 12 \sum_{i=1}^{\infty} \frac{1}{i^3}((1-1/2^n)^2 + i + 1 + 1 + 2) \leqslant 12\pi^2.$$

That is,

$$\left|\int_{\Omega^{k,1}_{b,a} \setminus I_{b_k+a}(x^1)} f_{b,a}(y)(U_{n_1}(t^1 - y^1)\, d\mu(y^1)\right| \leqslant 13\pi^2 \leqslant 130.$$

This completes the proof of Lemma 8. ∎

Let $x \in J^k_{b,a}$ and $t \in I_{b_k+3+j}(x^1) \times I_{b_k+3+a-j}(x^2)$ again, where $j \in \{d, d+1, \ldots, a-d\}$ and let $n_1 := 2^{b_k+j}$. Recall that

$$\Omega^{k,1}_{b,a} = \bigcup_{x \in J^k_{b,a}} I_{b_k}(x^1).$$

LEMMA 9. *We prove*

$$\left|\int_{\Omega^{k,1}_{b,a} \setminus I_{b_k+a}(x^1)} f_{b,a}(y) \sum_{i=1}^{2^{a-j}} \frac{\omega_{i2^{n_1}}(t^1 - y^1)}{i 2^{n_1}} \sum_{s=0}^{2^{n_1}-1} s\omega_s(t^1 - y^1)\, d\mu(y^1)\right| \leqslant (a-j).$$

Proof. For $z \in I$ we have

$$\sum_{s=0}^{2^{n_1}-1} s\omega_s(z) = \sum_{s=0}^{2^{n_1}-1} 2^{s-1}(D_{2^{n_1}}(z) - D_{2^{n_1}}(z + e_s)).$$

$t^1 \in I_{n_1}(x^1)$, $y^1 \in I_{b_k}(\tilde{x}^1)$ for some $\tilde{x} \in J^k_{b,a}$, $\tilde{x}^1 \neq x^1$ (otherwise $f_{b,a}(y) = 0$), consequently $t^1 - y^1 \notin I_{b_k}$ which gives $D_{2^{n_1}}(t^1 - y^1) = 0$. We also have that $D_{2^{n_1}}(t^1 - y^1 + e_s)$ can be different from zero only in the case when $s < b_k$ and for all $s < b_k$ there exists at most one $\tilde{x}^1 \in I$ for which there exists an $\tilde{x} = (\tilde{x}^1, \tilde{x}^2) \in J^k_{b,a}$ for which $t^1 - y^1 + e_s \in I_{b_k}(\tilde{x}^1)$. If the function $f_{b,a}$ is not the constant zero function on the set $I_{b_k}(\tilde{x}^1)$ then it differs from zero on $I_{b_k+a}(\tilde{x}^1) \subset I_{b_k}(\tilde{x}^1)$. That is,

$$\left| \int_{\Omega^{k,1}_{b,a} \setminus I_{b_k+a}(x^1)} f_{b,a}(y) \sum_{i=1}^{2^{a-j}} \frac{\omega_{i2^{n_1}}(t^1 - y^1)}{i2^{n_1}} \sum_{s=0}^{2^{n_1}-1} s\omega_s(t^1 - y^1)\, d\mu(y^1) \right|$$

$$\leqslant \sum_{s=0}^{b_k-1} 2^{s-1} \sum_{i=1}^{2^{a-j}} \frac{1}{i} 2^a \frac{1}{2^{b_k+a}} \leqslant (a-j).$$

∎

With the same conditions as in Lemma 8 and 9 we prove

COROLLARY 10.

$$\left| \int_{\Omega^{k,1}_{b,a} \setminus I_{b_k+a}(x^1)} f_{b,a}(y)(D_{2^n}(t^1 - y^1) + V_n(t^1 - y^1))\, d\mu(y^1) \right|$$

$$\leqslant 131(a-j).$$

Proof. $D_{2^{n_1}}(t^1 - y^1) = 0$ for $y^1 \in I_{b_k}(\tilde{x}^1)$, $\tilde{x}^1 \neq x^1$. Lemmas 8 and 9 with

$$\int_{I_{b_k+a}(\tilde{x}^1)} \omega_{i2^{n_1}}(t^1 - y^1)\, d\mu(y^1) = 0$$

(for $i \geqslant 2^{a-j}$) complete the proof of Corollary 10. ∎

Set

$$\Omega^{k,2}_{b,a} := \bigcup_{x \in J^k_{b,a}} I_{b_k}(x^2).$$

Recall that $n = (n_1, n_2) = (b_k + j, b_k + a - j) \in \mathbb{P}^2$.

COROLLARY 11.

$$\left| \int_{\bigcup \{ I^2_{b_k}(\tilde{x}) : \tilde{x} \in J^k_{b,a}, \tilde{x}^1 \neq x^1, \tilde{x}^2 \neq x^2 \}} f_{b,a}(y) \right.$$

$$\left. \times (D_{2^{n_1}}(t^1 - y^1) + V_{n_1}(t^1 - y^1)) \times (D_{2^{n_2}}(t^2 - y^2) + V_{n_2}(t^2 - y^2))\, d\mu(y) \right|$$

$$\leqslant \frac{(131a)^2}{2^a}.$$

Lemma 12.

$$\Bigg|\int_{I^2_{b_k+a}(x)} f_{b,a}(y)(D_{2^{n_1}}(t^1-y^1)+V_{n_1}(t^1-y^1))$$
$$\times(D_{2^{n_2}}(t^1-y^1)+V_{n_2}(t^1-y^1))\,d\mu(y)$$
$$-\int_{[\Omega^{k,1}_{b,a}\setminus I_{b_k+a}(x^1)]\times I_{b_k+a}(x^2)} f_{b,a}(y)(D_{2^{n_1}}(t^1-y^1)+V_{n_1}(t^1-y^1))$$
$$\times(D_{2^{n_2}}(t^1-y^1)+V_{n_2}(t^1-y^1))\,d\mu(y)$$
$$-\int_{I_{b_k+a}(x^1)\times[\Omega^{k,2}_{b,a}\setminus I_{b_k+a}(x^2)]} f_{b,a}(y)(D_{2^{n_1}}(t^1-y^1)+V_{n_1}(t^1-y^1))$$
$$\times(D_{2^{n_2}}(t^1-y^1)+V_{n_2}(t^1-y^1))\,d\mu(y)\Bigg|$$
$$\geqslant 2^{-3}.$$

Proof. We give a lower bound for

$$\Bigg|\frac{1}{2}\int_{I^2_{b_k+a}(x)} f_{b,a}(y)(D_{2^{n_1}}(t^1-y^1)+V_{n_1}(t^1-y^1)) \tag{4}$$
$$\times(D_{2^{n_2}}(t^1-y^1)+V_{n_2}(t^1-y^1))\,d\mu(y)$$
$$-\int_{[\Omega^{k,1}_{b,a}\setminus I_{b_k+a}(x^1)]\times I_{b_k+a}(x^2)} f_{b,a}(y)(D_{2^{n_1}}(t^1-y^1)+V_{n_1}(t^1-y^1))$$
$$\times(D_{2^{n_2}}(t^1-y^1)+V_{n_2}(t^1-y^1))\,d\mu(y)\Bigg|.$$

Since for $t^2\in I_{n_2+3}(x^2)$, $y^2\in I_{b_k+a}(x^2)$ then $t^2-y^2\in I_{n_2+3}$. This by Lemma 2 gives

$$(D_{2^{n_2}}(t^1-y^1)+V_{n_2}(t^1-y^1))\geqslant 2^{n_2-1}.$$

That is, (4) is not less than

$$\Bigg|\frac{1}{2}\int_{I_{b_k+a}(x^1)} f_{b,a}(y)(D_{2^{n_1}}(t^1-y^1)+V_{n_1}(t^1-y^1))\,d\mu(y^1)$$
$$-\int_{\Omega^{k,1}_{b,a}\setminus I_{b_k+a}(x^1)} f_{b,a}(y)(D_{2^{n_1}}(t^1-y^1)+V_{n_1}(t^1-y^1))\,d\mu(y^1)\Bigg|\,2^{n_2-1}\frac{1}{2^{b_k+a}}$$

$$\geqslant \left| \frac{1}{2} 2^{a+n_1} (-1)^{x_0^1+x_0^2+x_{b_k-1}^1+x_{b_k-1}^2} 2^{-b_k-a} \right.$$

$$+ \frac{1}{2} \int_{I_{b_k+a}(x^1)} f_{b,a}(y) \sum_{i=1}^{2^{a-j}-1} \frac{\omega_{i2^{n_1}}(t^1-y^1)}{i2^{n_1}} \sum_{s=0}^{2^{n_1}-1} s\omega_s(t^1-y^1)\, d\mu(y^1)$$

$$- \int_{\Omega_{b,a}^{k,1} \setminus I_{b_k+a}(x^1)} f_{b,a}(y) \sum_{i=1}^{2^{a-j}-1} \frac{\omega_{i2^{n_1}}(t^1-y^1)}{i2^{n_1}} \sum_{s=0}^{2^{n_1}-1} s\omega_s(t^1-y^1)\, d\mu(y^1)$$

$$\left. - \int_{\Omega_{b,a}^{k,1} \setminus I_{b_k+a}(x^1)} f_{b,a}(y)\, U_{n_1}(t^1-y^1)\, d\mu(y^1) \right| 2^{-j-1}.$$

For $y^1 \in I_{b_k+a}(x^1)$ we have $\sum_{s=0}^{2^{n_1}-1} s\omega_s(t^1-y^1) = 2^{n_1}(2^{n_1}-1)/2$. For $y^1 \in \Omega_{b,a}^{k,1} \setminus I_{b_k+a}(x^1)$ we have

$$\sum_{s=0}^{2^{n_1}-1} s\omega_s(t^1-y^1) = -\sum_{s=0}^{n_1-1} 2^{s-1} D_{2^{n_1}}(t^1-y^1+e_s).$$

As earlier, $D_{2^{n_1}}(t^1-y^1+e_s)$ can be different from zero only in the case when $s < b_k$ and for a given s there exists at most one $\tilde{x}^1 \in I$ for which there exists a $\tilde{x} = (\tilde{x}^1, \tilde{x}^2) \in J_{b,a}^k$, $\tilde{x}^1 \neq x^1$ for which $D_{2^{n_1}}(t^1-y^1+e_s) \neq 0$ ($y^1 \in I_{b_k}(\tilde{x}^1)$). If $y^1 \in I_{b_k+a}(\tilde{x}^1)$ for some $\tilde{x} \in J_{b,a}^k$ then $y_{b_k}^1 = \cdots = y_{b_k+a-1}^1 = 0$ which gives $\omega_{i2^{n_1}}(t^1-y^1) = \omega_{i2^{n_1}}(t^1)$ ($i = 1, \ldots, 2^{a-j}-1$). That is,

(5)

$$\frac{1}{2} \int_{I_{b_k+a}(x^1)} f_{b,a}(y) \sum_{i=1}^{2^{a-j}-1} \frac{\omega_{i2^{n_1}}(t^1-y^1)}{i2^{n_1}} \sum_{s=0}^{2^{n_1}-1} s\omega_s(t^1-y^1)\, d\mu(y^1)$$

$$- \int_{\Omega_{b,a}^{k,1} \setminus I_{b_k+a}(x^1)} f_{b,a}(y) \sum_{i=1}^{2^{a-j}-1} \frac{\omega_{i2^{n_1}}(t^1-y^1)}{i2^{n_1}} \sum_{s=0}^{2^{n_1}-1} s\omega_s(t^1-y^1)\, d\mu(y^1)$$

$$= \sum_{i=1}^{2^{a-j}-1} \frac{\omega_{i2^{n_1}}(t^1)}{i2^{n_1}} \left(\frac{1}{2} \int_{I_{b_k+a}(x^1)} 2^a (-1)^{x_0^1+x_0^2+x_{b_k-1}^1+x_{b_k-1}^2} \frac{2^{n_1}(2^{n_1}-1)}{2}\, d\mu(y^1) \right.$$

$$\left. + \sum_{s=0}^{b_k-1} 2^{s-1} \int_{I_{b_k+a}(x^1+e_s)} f_{b,a}(y)\, D_{2^{n_1}}(t^1-y^1+e_s)\, d\mu(y^1) \right).$$

By Lemma 2 or more exactly, by the method with which we proved Lemma 2 we have the following lower bound for the absolute value of (5)

$$\frac{1}{2} \frac{1}{2^{n_1}} \left(\frac{1}{4} \frac{2^{n_1}(2^{n_1}-1)}{2^{b_k}} - 2^{b_k} \frac{2^a 2^{n_1}}{2^{b_k+a}} \right) \geqslant 2^{j-4} - \frac{1}{2}.$$

The sign of (5) is $(-1)^{x_0^1+x_0^2+x_{b_k-1}^1+x_{b_k-1}^2}$. Consequently,

$$(4) \geqslant \left[\tfrac{1}{2}2^j+2^{j-4}-\tfrac{1}{2}-\left|\int_{\Omega_{b,a}^{k,1}\setminus I_{b_k+a}(x^1)} f_{b,a}(y)\, U_{n_1}(t^1-y^1)\, d\mu(y^1)\right|\right]2^{-j-1}$$
$$\geqslant (2^{j-1}-130)\, 2^{-j-1}\geqslant \tfrac{1}{8}.$$

(In the last inequality we used Lemma 8 and $j \geqslant d \geqslant 10$.) Applying the above written in the proof of this lemma for the other coordinate (considering that the signs of the two terms—the absolute value of the first one is (4)—are the same) we complete the proof. ∎

At last with the help of Corollary 11 and Lemmas 12, 6, and 7 the proof of Lemma 5 is complete. ∎

Next we turn our attention to the construction of the counterexample function. Define $\beta_n, a_n, \delta_n \in \mathbb{P}$ in the following way $\beta_0 = a_0 = \delta_0 := 5d$. For $n \in \mathbb{N}$ let

$$\beta_n > \sum_{k=0}^{n-1} \beta_k 2^{a_k}$$

$$\delta_n := \left[\sup\left\{t \in \mathbb{R} : \delta(t) > \frac{1}{2^n \beta_n}\right\}\right]+1$$

$$\text{(if } \{t\colon \delta(t) > 1/(2^n\beta_n)\} = \varnothing, \text{ then } \delta_n := 5d)$$

$$2^{a_n} > \delta_n, 2\beta_n, 2^n, \qquad \sum_{n=0}^{\infty} \frac{\beta_n}{a_n} < \infty.$$

Define the function $F\colon I\times I^2 \to \mathbb{R}$ as

$$F(u, x) = \sum_{n=0}^{\infty} r_n(u)\, \beta_n f_n(x) := \sum_{n=0}^{\infty} r_n(u)\, \beta_n f_{b, a_n}(x).$$

Note that in the definition of $f_{b,a_n}(x)$, $b_0 := 2$ and $b_k > 4(b_{k-1}+a_n+1)$ for all $k \in \mathbb{N}$.

At first we prove

LEMMA 13. $\int_{I^2} |F(u, x)|\, log^+(|F(u, x)|)\, \delta(|F(u, x)|)\, d\mu(x) \leqslant 16.$

Proof. Set

$$H_n := \{x \in I^2 : f_n(x) \neq 0, f_{n+j}(x) = 0\ (j \in \mathbb{N})\} \qquad (n \in \mathbb{P})$$

and $H_{-1} := \{x \in I^2 : f_j(x) = 0 \ (j \in \mathbb{P})\}$. The definiton of f_{b, a_n}, (a_n) gives

$$\begin{aligned}
&\mu(\{x \in I^2 : f_{n+j}(x) = 0 \ (j \in \mathbb{N})\}) \\
&\quad \geqslant 1 - \mu\left(\bigcup_{k>n} \{x \in I^2 : f_k(x) \neq 0\} \right) \geqslant 1 - \sum_{k>n} \mu(\{x \in I^2 : f_k(x) \neq 0\}) \\
&\quad \geqslant 1 - \sum_{k>n} \frac{1}{2^{a_k}(a_k/2+1)} \geqslant 1 - \sum_{k>n} \frac{1}{2^k}.
\end{aligned}$$

This follows $\bigcup_{n=-1}^{\infty} \{x \in I^2 : f_{n+j}(x) = 0 \ (j \in \mathbb{N})\} = I^2$ (neglecting a set of measure zero). Thus, $\bigcup_{n=-1}^{\infty} H_n = I^2$ (neglecting a set of measure zero). Corresponding to this argument if $x \in H_n$ $(n \in \mathbb{P})$ then

$$\begin{aligned}
|F(u, x)| &\leqslant \sum_{k=0}^{n-1} \beta_k 2^{a_k} + \beta_n 2^{a_n} \\
&\leqslant \beta_n + \beta_n 2^{a_n} \leqslant 2\beta_n 2^{a_n} = |\beta_n 2 f_{b, a_n}(x)|, \\
|F(u, x)| &\geqslant \beta_n 2^{a_n} - \sum_{k=0}^{n-1} \beta_k 2^{a_k} \\
&\geqslant \beta_n 2^{a_n} - \beta_n \geqslant \tfrac{1}{2} \beta_n 2^{a_n} \\
&= \tfrac{1}{2} |\beta_n f_{b, a_n}(x)|.
\end{aligned}$$

Moreover, for $x \in H_n$ we have $|F(u, x)| \geqslant \frac{1}{2} \beta_n 2^{a_n} \geqslant 2^{a_n} > \delta_n$, which gives

$$\delta(|F(u, x)|) \leqslant \frac{1}{2^n \beta_n}.$$

Consequently, by Lemma 4

$$\begin{aligned}
&\int_{H_n} |F(u, x)| \log^+(|F(u, x)|) \, \delta(|F(u, x)|) \, d\mu(x) \\
&\quad \leqslant \int_{H_n} 2 \, |\beta_n f_{b, a_n}(x)| \log^+(2\beta_n |f_{b, a_n}(x)|) \frac{1}{2^n \beta_n} \\
&\quad \leqslant \int_{H_n} 2 \, |\beta_n f_{b, a_n}(x)| \log^+(|f_{b, a_n}(x)|^2) \frac{1}{2^n \beta_n} \\
&\quad \leqslant \frac{4}{2^n} \int_{I^2} |f_{b, a_n}(x)| \log^+(|f_{b, a_n}(x)|) \, d\mu(x) \leqslant \frac{8}{2^n}.
\end{aligned}$$

Since for $x \in H_{-1}$ we have $F(u, x) = 0$, then we get

$$\int_{I^2} |F(u,x)| \log^+(|F(u,x)|)\, \delta(|F(u,x)|)\, d\mu(x)$$

$$\leqslant \sum_{n \in \mathbb{P}} \int_{H_n} |F(u,x)| \log^+(|F(u,x)|)\, \delta(|F(u,x)|)\, d\mu(x) \leqslant 16.$$

∎

The following lemma can be found in the paper of Stein [Ste] or in the book of Zygmund [Z, p. 213, I].

Lemma 14. *Let $E \subset I$ be a measurable set with positive measure. Then there exists an $N \in \mathbb{P}$ and a constant $A \in \mathbb{R}$ so that*

$$\left(\sum_{n \geqslant N} |\gamma_n|^2 \right)^{\frac{1}{2}} \leqslant A \text{ ess sup}_{u \in E} |F(u)|$$

for all

$$\sum_{n=0}^{\infty} |\gamma_n|^2 < \infty, \qquad F(u) = \sum_{n=0}^{\infty} \gamma_n r_n(u)$$

Rademacher series.

Now, we are ready to prove Theorem 1.

Proof of Theorem 1. Suppose on the contrary that there exists a measurable function δ: $[0, +\infty) \to [0, +\infty)$, $\lim_{t \to \infty} \delta(t) = 0$ such as that for all functions $f \in L \log^+ L\delta(L)$ with the property

$$\hat{f}(n_1, n_2) = 0 \qquad (n_1 n_2 = 0, n \in \mathbb{P}^2)$$

$\sup_{n \in \mathbb{P}^2} |d_n(If)| < +\infty$ on a positive measure subset of I^2. Consequently, we have

$$\sup_{n \in \mathbb{P}^2} |T_n F(u, x))| < +\infty$$

on a positive measure subset of I^2 with respect to each $u \in I$. Let $m \in \mathbb{P}^2$. Since T_m is a linear operator then for all $u \in I$, $x \in I^2$, and $K \in \mathbb{P}$

$$T_m \left(\sum_{n=0}^{K} r_n(u)\, \beta_n f_{b, a_n} \right)(x) = \sum_{n=0}^{K} r_n(u)\, \beta_n (T_m f_{b, a_n})(x).$$

The operator T_m is of type (1, 1) (see, e.g., [SW]) which gives

$$\sum_{n=0}^{\infty} \|\beta_n (T_m f_{b,a_n})\|_1 \leqslant C \sum_{n=0}^{\infty} \|\beta_n f_{b,a_n}\|_1$$
$$\leqslant C \sum_{n=0}^{\infty} \beta_n 2^{a_n} \mu(f_{b,a_n} \neq 0) = C \sum_{n=0}^{\infty} \beta_n 2^{a_n} \frac{1}{2^{a_n}(a_n/2+1)}$$
$$\leqslant C \sum_{n=0}^{\infty} \frac{\beta_n}{a_n} < \infty.$$

That is,

$$\sum_{n=0}^{\infty} |\beta_n (T_m f_{b,a_n})|^2 \leqslant \sum_{n=0}^{\infty} |\beta_n (T_m f_{b,a_n})| < \infty$$

a.e. This implies that the function

$$g(u, x) := \sum_{n=0}^{\infty} r_n(u)\, \beta_n (T_m f_{b,a_n})(x)$$

exists and is finite for all $u \in I$ and a.e. x. That is, for all $K \in \mathbb{P}$

$$\left\| T_m F(u, .) - \sum_{n=0}^{\infty} r_n(u)\, \beta_n (T_m f_{b,a_n})(.) \right\|_1$$
$$\leqslant \left\| T_m \left(\sum_{n=K+1}^{\infty} r_n(u)\, \beta_n f_{b,a_n})(.) - \sum_{n=K+1}^{\infty} r_n(u)\, \beta_n (T_m f_{b,a_n})(.) \right\|_1$$
$$\leqslant C \sum_{n=K+1}^{\infty} \beta_n \|f_{b,a_n})(.)\|_1 \leqslant C \sum_{n=K+1}^{\infty} \frac{\beta_n}{a_n}$$

tends to zero as K tends to infinity. That is,

$$T_m F(u, x) = T_m \left(\sum_{n=0}^{\infty} r_n(u)\, \beta_n f_{b,a_n} \right)(x) = \sum_{n=0}^{\infty} r_n(u)\, \beta_n (T_m f_{b,a_n})(x)$$

for all $m \in \mathbb{P}^2$, $u \in I$, and a.e. $x \in I^2$. This gives

$$+\infty > \sup_{m \in \mathbb{P}^2} |T_m F(u, x)| \geqslant \left| \sum_{n=0}^{\infty} r_n(u)\, \beta_n (T_m f_{b,a_n})(x) \right|$$

for all $m \in \mathbb{P}^2$ on a positive measure subset of I^2 with respect to each $u \in I$. Thus, there exists constant $A > 0$ such that for a positive measure of $(u, x) \in I \times I^2$

$$\sup_{m \in \mathbb{P}^2} |T_m F(u, x)| < A.$$

Denote by this set of pairs (u, x) by E. $E_x := \{u \in I : (u, x) \in E\}$. For a positive measure of $x \in I^2$ we have that the measure of E_x is greater than zero. By Lemma 14 it follows the existence of a constant A_x and $N_x \in \mathbb{P}$ such that for all $m \in \mathbb{P}^2$

$$\left(\sum_{n=N_x}^{\infty} \beta_n^2 \, |(T_m f_{b, a_n})(x)|^2 \right)^{1/2} \leqslant A_x \text{ ess sup}_{u \in E_x} |T_m F(u, x)| \leqslant A_x A.$$

The construction of $G_{b, a, \circ}$ gives that $\mu(\limsup_n G_{b, a_n, \circ}) = 1$. We give a sketch of the proof of this. Take $d_n \in \mathbb{P}$ such that $\mu(I^2 \setminus \bigcup_{k<d_n} \Omega^k_{b, a_n}) < \frac{1}{2^n}$. Thus, $\mu(\limsup_n \bigcup_{k \geqslant d_n} \Omega^k_{b, a_n}) = 0$. We also have $\mu(\limsup_n \Omega^0_{b, a_n}) = 0$. That is, it can be supposed that an $x \in I^2$ is not in $\cup_{1 \leqslant k < d_n} \Omega^k_{b, a_n}$ for only a finite numbers of n. Define the sequence of natural numbers (n_j) in a way that $b_1 = b_1(a_{n_j})$ is greater than the greatest index which occurs related the dyadic rectangles establishing $\Omega^k_{b, a_{n_i}}$ $(1 \leqslant k \leqslant d_{n_i}, i < j)$. By this we have $\mu(\bigcap_{i=1}^{j} \bigcup_{1 \leqslant k < d_{n_i}} (\Omega^k_{b, a_{n_i}} \setminus \Omega^k_{b, a_{n_i}, \circ})) \leqslant \prod_{i=1}^{j} \mu(\bigcup_{1 \leqslant k < d_{n_i}} (\Omega^k_{b, a_{n_i}} \setminus \Omega^k_{b, a_{n_i}, \circ})) \leqslant (1 - \frac{1}{2^7})^j$. This certainly implies that $\mu(\liminf_n (I^2 \setminus G_{b, a_n, \circ})) = 0$. $n \in \mathbb{P}$ for which $x \in G_{b, a_n, \circ}$ and consequently, by Lemma 5 there is an $m \in \mathbb{P}^2$ such that $|T_m f_{b, a_n})(x)| \geqslant 2^{-4}$. Since $n \geqslant N_x$ can be supposed we have

$$\beta_n 2^{-4} \leqslant \left(\sum_{n=N_x}^{\infty} \beta_n \, |(T_m f_{b, a_n})(x)|^2 \right)^{1/2} \leqslant A_x A$$

for an infinite number of $n \in \mathbb{P}$. This is a contradiction. That is, the proof of Theorem 1 is complete. ∎

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